VECTOR CALCULUS

1. $\nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$
2. If ϕ is a scalar point function then,

$$\nabla\phi = \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right)\phi$$

$$= \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

$\nabla\phi$ is called the gradient of ϕ (ϕ is a scalar function) and is denoted by grad ϕ

i. e., grad $\phi = \nabla\phi$

3. i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

ii) $\nabla(fg) = f\nabla g + g\nabla f$

iii) $\nabla C = 0$ where C is a constant

iv) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $|\vec{r}| = r$ then,

$$\nabla r = \frac{\bar{r}}{r} = \hat{r}$$

v) $\nabla r^n = nr^{n-2}\vec{r}$

vi) $\nabla f(r) = f'(r)\left(\frac{\bar{r}}{r}\right)$

vii) $\nabla(\log r) = \frac{\bar{r}}{r^2}$

viii) $\nabla f(r) \times \bar{r} = 0$

4. The directional derivative of a scalar point function ϕ at the point (x, y, z) in the direction of a given vector $\vec{a}$ is $\nabla\phi.\frac{\bar{a}}{|\bar{a}|}$
5. The unit normal to the surface $\phi = 0$ at (x_1, y_1, z_1) is

$$\bar{n} = \frac{\nabla\phi(x_1, y_1, z_1)}{|\nabla\phi(x_1, y_1, z_1)|}$$

ie., $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

6. The directional derivative at a point P is maximum in the direction of the normal at P. The maximum directional derivative at P = $|\nabla\phi|$.
7. Angle between the surfaces $\phi_1 = 0$ and $\phi_2 = 0$ at (x_0, y_0, z_0) is $\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$
8. Equation of tangent plane and normal plane for the surface $\phi = 0$ at a.

Equation of the tangent plane is $(\bar{r} - \bar{a}).\nabla\phi = 0$

Equation of the normal plane is $(\bar{r} - \bar{a}) \times \nabla\phi = 0$

9. Let $\bar{F}$ be a vector valued function. Then,

div $\bar{F} = \nabla.\bar{F}$

curl $\bar{F} = \nabla\times\bar{F}$

10. If $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ Then,

div $\bar{F}$

$$= \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right).\left(F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl $\bar{F} = \nabla \times \bar{F}$

$$= \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

11. div $\bar{F}$ is a scalar.

curl $\bar{F}$ is a vector quantity.

12. A vector F is said to be solenoidal if $\nabla.\bar{F} = 0$

i. e., div $\bar{F} = 0$

13. A vector $\bar{F}$ is said to be irrotational if curl $\bar{F} = 0$

i.e., $\nabla\times\bar{F} = 0$

14. i) div $\left(\frac{\bar{r}}{r}\right) = \frac{2}{r}$

ii) div $(r^n\bar{r}) = (n + 3)r^n$

iii) div $(r^n(\bar{a} \times \bar{r})) = 0$

Where $\bar{a}$ is a constant.

iv) $\nabla\left(\frac{f(r)}{r}\bar{r}\right) = \frac{1}{r^2}\frac{d}{dr}(r^2 f(r))$

15. i) $\nabla\times\bar{r} = 0$

ii) $\nabla\times(f(r)\bar{r}) = 0$

iii) $\nabla\times(r^n\bar{r}) = 0$

16. If ϕ is a scalar function and $\bar{F}$ is a vector function, then,

i) $\nabla.\nabla\phi = \nabla^2\phi$

ii) $\nabla\times(\nabla\phi) = 0$

i.e., curl grad $\phi = 0$

iii) $\nabla.(\nabla\times\bar{F}) = 0$

iv) curl curl curl curl $\bar{F} = \nabla^4\bar{F}$

v) $\nabla\times(\nabla r^n) = 0$

vi) $\nabla.(\nabla r^n) = n(n+1)r^{n-2}$

vii) $\nabla.(\phi\bar{F}) = (\nabla\phi).\bar{F} + \phi(\nabla.\bar{F})$

viii) $\nabla\times(\phi\bar{F}) = (\nabla\phi)\times\bar{F} + \phi(\nabla\times\bar{F})$

ix) $\nabla.(\bar{F}\times\bar{G}) = \bar{G}.(\nabla\times\bar{F}) - \bar{F}.(\nabla\times\bar{G})$

i.e., div $(\bar{F}\times\bar{G}) = \bar{G}$. curl $\bar{F}$ - $\bar{F}$. curl $\bar{G}$

$\nabla\times(\nabla\times\bar{F}) = \nabla(\nabla.\bar{F}) - \nabla^2\bar{F}$

i.e., curl curl $\bar{F} = \nabla$ div $\bar{F} - \nabla^2\bar{F}$

Gauss Divergence Theorem:

Let $\bar{F}$ be a vector point function, finite and differentiable in the region R bounded by a closed surface S, then the surface integral of the normal component of $\bar{F}$ taken over S is equal to the integral of divergence of $\bar{F}$ taken over V.

i.e., $\iint_S F.\hat{n}ds = \iiint_V \nabla.\bar{F}dv$

1. **Deductions from Gauss divergence theorem:**

i) $\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv = \iint_S (\phi\nabla\psi - \psi\nabla\phi).nds$ where, ϕ and ψ are scalar point functions.

ii) $\iint_V \nabla\phi dv = \iint_S \hat{n}.\phi ds$

iii) $\iiint_V \nabla\times F\,dv = \iint \hat{n}.\times\bar{F}ds$

2. **Cartesian form of Gauss divergence theorem:**

Let $\hat{n} = \cos\alpha\bar{i} + \cos\beta\bar{j} + \cos\gamma\,\bar{k}$

where, α, β and γ are the angles which $\hat{n}$ makes with x, y, z axes respectively.

Let, $\bar{F} = F_1 i + F_2 j + F_3 k$

Then,

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv \quad dx\,dy\,dz$$

$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,ds$$

$$= \iint_S F_1\,dydz + F_2\,dzdx + F_3\,dxdy$$

3. If S is a closed surface then

$$\iint_S \bar{r}.\hat{n}\,ds = 3V$$

$$\iint_S \text{curl}\,\bar{F}.\hat{n}\,ds = 0$$

$$\iint_S \nabla r^2.\hat{n}\,ds = 6V$$

$$\iint_S \frac{\bar{r}.\hat{n}}{r^2}\,ds = \iiint_V \frac{dv}{r^2}$$

Stokes Theorem:

Let S be an open surface bounded by a closed curve C. Let $\bar{F}$ be a vector point function defined on the surface S and $\hat{n}$ is a unit outward normal at any point P on S.

Then, $\int_C F.d\bar{r} = \iint_S (\nabla\times\bar{F}).\bar{n}\,ds$

Deductions from Stokes Theorem:

i) $\int_C \bar{r}.d\bar{r} = 0$

ii) $\int_C \phi\nabla\psi.dr = -\int_C \psi\nabla\phi.dr$

iii) $\int_C \phi\nabla\phi\,dr = 0$

Green's theorem in plane:

Let R be a closed curve C. Let M and N be continuous functions of x and y having continuous partial derivatives in R. Then,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

Where C is traversed in the anti-clock wise direction.

Green's theorem in plane (vector form)

Let $\bar{F} = M\,\bar{i} + N\,\bar{j}$

$\bar{r} = x\,\bar{i} + y\,\bar{j}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

Then, Green's theorem in vector form is

$$\oint_C \bar{F}\,.\,d\bar{r} = \iint_R (\nabla \times \bar{F})\,.\,KdR$$

Result:

Area bounded by any closed curve C is given by

$$\frac{1}{2}\oint_C (xdy - ydx)$$