TRANSFORMS

Laplace Transform:

Definition:

If a function f(t) is defined for all positive values of t and $\int_0^\infty e^{-st} f(t)\, dt$ exists then the Laplace transform of f(t) is defined as

L [f(t)] = F(s)

$= \int_0^\infty e^{-st} f(t)\, dt$

Properties:

1. L [f(t) ± g(t)] L(f(t)) ± L(g(t))
2. L(Cf(t)) = CL(f(t))
   where C is a constant
3. L(f'(t)) = s L(f (t)) - f(0)
4. $L(f''(t)) = s^2 L(f(t)) - Sf(0) - f'(0)$
   $\ldots -f^{n-1}(0)$
5. $L((t)) = s^n L(f(t)) - S^{n-1} f(0) - S^{n-2} f'(0) - \ldots\ldots - f^{n-1}(0)$
6. If L(f(t)) = F(s) then,
   i) $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s F(s)$
   ii) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s F(s)$
7. **Some important formulae:**

S.No.	f(t)	F(s) = L(f(t))
1.	1	$\frac{1}{s}$
2.	t	$\frac{1}{s^2}$
3.	t^2	$\frac{2}{s^3}$
4.	t^3	$\frac{3!}{s^4} = \frac{6}{s^4}$
5.	t^n	$\frac{n!}{s^{n+1}}$
6.	$t^{1/2}$	$\frac{\sqrt{n}}{2s^{3/2}}$
7.	$t^{-1/2}$	$\frac{\sqrt{\pi}}{2s^{1/2}}$
8.	sin at	$\frac{a}{s^2 + a^2}$
9.	cos at	$\frac{s}{s^2 + a^2}$
10.	sinh at	$\frac{a}{s^2 - a^2}$
11.	cosh at	$\frac{s}{s^2 - a^2}$
12.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$
13.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2 + b^2}$
14.	e^{at}	$\frac{1}{s-a}$

8. If L f(t) = F (s) then, L f(at) = $\frac{1}{a} F\left(\frac{s}{a}\right)$
9. L (e^{-at} f(t)) = F (s+a) where L (f(t)) = F(s)
10. If L(f(t)) = F(s), then
    i) L (t f(t)) = $-\frac{d}{ds}$(F(s))
    ii) L (t^n f(t)) = $(-1)^n \frac{d^n}{ds^n}$(F(s))
11. If $\frac{f(t)}{t}$ has a limit as t ⟶ 0 and if L(f(t)) = F (s).
    Then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\, ds$

Inverse Laplace Transforms:

If $L(f(t)) = F(s)$ then,
$f(t) = L^{-1}(F(s))$

S.No.	F(s)	$L^{-1}(F(s)) = f(t)$
1.	$\frac{1}{s-a}$	e^{at}
2.	$\frac{s}{s^2+a^2}$	cos at
3.	$\frac{a}{s^2+a^2}$	sin at
4.	$\frac{s}{s^2-a^2}$	cosh at
5.	$\frac{a}{s^2-a^2}$	Sinh at
6.	$\frac{1}{s}$	1
7.	$\frac{1}{s^2}$	t
8.	$\frac{1}{s^3}$	$\frac{t^2}{2}$
9.	$\frac{1}{s^4}$	$\frac{t^3}{6}$
10.	$\frac{n!}{s^{n+1}}$	t^n
11.	$\frac{1}{(s-a)^2}$	te^{at}
12.	$\frac{2}{(s-a)^3}$	t^2e^{at}
13.	$\frac{n!}{(s-a)^{n+1}}$	t^ne^{at}
14.	$\frac{b}{(s-a)^2+b^2}$	$e^{at}\sin bt$
15.	$\frac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$
16.	$\frac{2as}{(s^2+a^2)^2}$	t sin at
17.	$\frac{s^2-a^2}{(s^2+a^2)^2}$	t cos at
18.	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
19.	$\frac{1}{s^{3/2}}$	$2\sqrt{\frac{t}{\pi}}$
20.	$\frac{1}{(s-a)^n}\ (n = 1, 2,)$	$\frac{1}{(n-1)!}t^{n-1}e^{at}$
21.	$\frac{1}{(s-a)(s-b)}\ (a \neq b)$	$\frac{1}{(a-b)}[e^{at} - e^{bt}]$
22.	$\frac{s}{(s-a)(s-b)}\ (a \neq b)$	$\frac{1}{(a-b)}[ae^{at} - be^{bt}]$
23.	$\frac{1}{s(s^2+a^2)}$	$\frac{1}{a^2}(1-\cos at)$
24.	$\frac{1}{s^2(s^2+a^2)}$	$\frac{1}{a^3}(at - \sin at)$
25.	$\frac{1}{(s^2+a^2)^2}$	$\frac{1}{2a^3}(\sin at - at\cos at)$
26.	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at - at\cos at)$
27.	$\frac{s}{(s^2+a^2)(s^2+a^2)}\ (a \neq b)$	$\frac{1}{b^2-a^2}(\cos at - \cos bt)$
28.	$\frac{e^{-as}}{s}$	$u(t-a)$
29.	e^{-as}	$\delta(t-a)$
30.	1	$\delta(t)$ unit step

		function
31.	$\log \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$

Important Theorems:

Let $L^{-1}(F(s)) = f(t)$ then,

1. $L^{-1}[F(s+a)] = e^{-at} L^{-1}(F(S))$
2. $L^{-1}[F(s)] = \frac{-1}{t} L^{-1}(F(s))$
3. $L^{-1}(S F(s)) = \frac{d}{dt} L^{-1}[F(s)]$

   If $L^{-1}(F(S)) \longrightarrow 0$ as $t \longrightarrow 0$
4. $L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^1 L^{-1}(F(s))dt$

Fourier Transforms:

1. Infinite Fourier Transform (Complex form) :

   $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

   $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

   (Inversion formula)
2. Fourier Cosine Transform:

   $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$

   $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_C(s) \cos sx\, ds$

   (Inversion formula)
3. Fourier Sine Transform:

   $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$

   $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_S(S) \sin sx\, ds$

   (Inversion formula)

Properties of Fourier Transforms:

1. $F[a f(x) + b g(x)] = aF(f(x)) + bF(g(x))$
2. If $F(f(x)) = F(s)$, then

   $F[f(x-a)] = e^{isa}F(s)$
3. If $F[f(x)] = F(s)$, then

   $F[f(ax)] = \frac{1}{|a|} F\left(\frac{s}{a}\right), a \neq 0$

   (Change of scale property)
4. $F[e^{iax} f(x)] = F(s+a)$
5. Modulation theorem:

   If $F(f(x)) = F(s)$, then

   $F[f(x) \cos ax] = \frac{1}{2}[F(s-a) + F(s+a)]$
6. If $F[f(x)]$ $F(s)$, then

   $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$
7. If $F[f(x)] = F(s)$ then

   $F[f'(x)] = -is F(s)$ if $f(x) \longrightarrow 0$

   as $x \longrightarrow \pm\infty$
8. $F\left[\int_a^x f(x)\, dx\right] = \frac{F(s)}{(-is)}$
9. The convolution of two functions f(x) and g(x) is defined as $f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$
10. The Fourier transform of the convolution of f(x) and g(x) is the product of their Fourier transforms. That is, $F[f(x) * g(x)] = F(s).g(s)$ $= F[f(x)] . F[g(x)]$
11. Parseval's identity:

    $$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$
12. Let $F_C(s)$, $F_S(s)$ be the Fourier cosine and sine transforms of f(x) respectively. Then,

    $F_C[a f(x) + bg(x)] = aF_C(f(x)) + bF_C(g(x))$

    $F_S[a f(x) + bg(x)] = aF_S(f(x)) + bF_S(g(x))$

    $F^s[f(x) \sin ax] = \frac{1}{2}[F_c(s-a) - F_C(s+a)]$

    $F_s[f(x) \cos ax] = \frac{1}{2}[F_S(s+a) + F_s(s-a)]$

    $F_C[f(x) \sin ax] = \frac{1}{2}[F_S(a+s) + F_S(a-s)]$

    $F_C[f(x) \cos ax] = \frac{1}{2}[F_C(s+a) + F_C(s-a)]$

    $\int_0^{\infty} f(x)\, g(x)\, dx = \int_0^{\infty} F_C(s)\, G_C(s)\, ds$

    $\int_0^{\infty} f(x)\, g(x)\, dx = \int_0^{\infty} F_S(s)\, G_S(s)\, ds$

    $\int_0^{\infty} |f(x)|^2 dx = \int_0^{\infty} |F_C(s)|^2 ds$

    $= \int_0^{\infty} |F_C(s)|^2 ds$

    Relationship between Fourier and Laplace Transforms.

Consider f(t) = $\begin{cases} e^{-xt}g(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$

then, F(f(t)) = $\frac{1}{2\pi}$ L (g(t))

Z - Transform:

If the function u_n is defined for discrete values (n = 0, 1, 2, ……) and u_n = 0 for n < 0, then its Z - transform is defined as

$Z(u_n) = \bar{u}(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

The inverse Z - transform is defined as $Z^{-1}(\bar{u}(z)) = u_n$

Some Standard Z - Transforms:

S.No.	U_n	$Z(U_n)$
1.	a^n	$\frac{z}{z-a}$
2.	n^p	$-z\frac{d}{dz}Z(n^{P-1})$ where P is a positive integer
3.	1	$\frac{z}{z-1}$
4.	N	$\frac{2}{(z-1)^2}$
5.	n^2	$\frac{z^2+z}{(z-1)^3}$
6.	n^3	$\frac{z^3+4z^2+z}{(z-1)^4}$
7.	n^4	$\frac{z^4+11z^3+11z^2+z}{(z-1)^5}$
8.	$n\,a^n$	$\frac{az}{(z-a)^2}$
9.	n^2a^n	$\frac{az^2+a^2z}{(z-a)^3}$
10.	$\cos n\theta$	$\frac{z(\bar{n}-\cos\theta)}{z^2-2z\cos\theta+1}$
11.	$\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
12.	$a^n\cos n\theta$	$\frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$
13.	$A^n\sin n\theta$	$\frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$
14.	$\frac{1}{n!}$	$\frac{1}{e^z}$
15.	$\frac{1}{(n+1)!}$	$z\left(\frac{1}{e^z}-1\right)$
16.	$\frac{1}{(n+2)!}$	$z^2\left(\frac{1}{e^z}-1-\frac{1}{z}\right)$
17.	$z[(-1)^n]$	$\frac{z}{z+1}$
18.	$\cosh n\theta$	$\frac{z(z-\cos h\theta)}{z^2-2z\cos h\theta+1}$

Damping Rule:

If $Z(u_n) = \bar{u}(z)$ then

$Z(a^{-n}u_n) = \bar{u}(az)$ and

$Z(a^n u_n) = \bar{u}\left(\frac{z}{a}\right)$

Shifting theorem:

Let $Z(u_n) = \bar{u}(z)$ and k > 0 then,

i) $Z(u_{n-k}) = z^{-k}\bar{u}(z)$

ii) $Z(u_{n+k}) = z^k[\bar{u}(z) - u_0 - u_1z^{-1} - u_2z^{-2} - \cdots - u_{k-1}z^{-(k-1)}$

Putting k = 1, 2, 3 in the above theorem.

iii) $Z(u_{n+1}) = Z(\bar{u}(z) - u_0)$

iv) $Z(u_{n+2}) = z^2[\bar{u}(z) - u_0 - u_1z^{-1}]$

v) $Z(u_{n+3}) = z^3[\bar{u}(z) - u_0 - u_1z^{-1} - u_2z^{-2}]$

Initial value theorem:

Let $Z(u_n) = \bar{u}(z)$, then

$u_0 = \lim_{z\to 0}\bar{u}(z)$

Final value theorem:

If $Z(u_n) = \bar{u}(z)$ then,

$\lim_{n\to\infty} u_n = \lim_{z\to 1}(z-1)\,\bar{u}(z)$

Convolution theorem:

If $Z^{-1}(\bar{u}(z)) = u_n$ and

$\overline{Z_1}(\bar{v}(z)) = v_n$ then

$Z^{-1}[\bar{u}(z), \bar{v}(z)] = \sum_{m=0}^{\infty} u_m\, v_{n-m} = u_n * v_n$