Functions of Complex Variables and Complex Integration

1. **Analytic function:**
If a function f(z) has a derivative at z_o and at every point in some neighbourhood of z_o, then f(z) is said to be analytic at z_o. f(z) is said to be analytic in a Domain D, if it is analytic at every point of D.

2. **Cauchy - Riemann Equations:**
The necessary conditions for a complex function f(z) = u (x, y) + iv (x, y) to be analytic are
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$
i.e., $u_x = v_y$; $v_x = -u_y$

3. **Sufficient condition for f(z) to be analytic**
The function f(z) = u(x, y) + iv(x, y) is analytic in a domain D if
i) u(x, y) and v(x, y) are differentiable in D and $u_x = v_y$ and $u_x = -v_x$
ii) The partial derivatives u_x, u_y, v_x and v_y are all continuous in D.

4. **Polar-form of Cauchy - Riemann equations**
Let $f(z) = P(r,\theta) + iQ(r,\theta)$
Then,
$$\frac{\partial P}{\partial r} = \frac{1}{r}\frac{\partial Q}{\partial \theta} \text{ and } \frac{\partial Q}{\partial r} = -\frac{1}{r}\frac{\partial P}{\partial \theta}$$

5. If w = f(z) is analytic in a domain D then,
i) $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$
ii) $\frac{\partial^2 w}{\partial z\,\partial \bar{z}} = 0$
iii) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

6. If f(z) = P(r, θ) + iQ (r, θ) is analytic, then
$$f'(z) = e^{-I\theta}\left[\frac{\partial P}{\partial r} + i\frac{\partial Q}{\partial r}\right]$$
$$= \frac{1}{r}e^{-I\theta}\left[\frac{\partial Q}{\partial \theta} + i\frac{\partial P}{\partial \theta}\right]$$

7. **Construction of Analytic function Milne - Thompson method:**
If Real part u(x, y) of an analytic function f(z) is given then,
$$f(z) = \int\left\{\frac{\partial u(z,0)}{\partial x} - i\frac{\partial u(z,0)}{\partial y}\right\}dz + C$$
If Imaginary part v(x, y) of f(z) is given, then,
$$f(z) = \int\left\{\frac{\partial v(z,0)}{\partial y} + i\frac{\partial v(z,0)}{\partial x}\right\}dz + C$$

8. **Some Results:**
i) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$
ii) If f(z) is analytic and f(z) is constant then
a) $f'(z) = 0$ everywhere
b) $\overline{f(z)}$is also analytic
c) R(f(x)) is a constant
d) | f(z) | is constant
iii) $f(z) = \bar{z}$is nowhere differentiable.
iv) $f(z) = |z|^2$ Is differentiable only at the origin.
v) Both the real part and the imaginary parts of any analytic function satisfies Laplace equation.
i.e., If f(z) = u + iv is analytic then,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$
vi) Any function which has continuous second order partial derivatives and which satisfies Laplace equation is called Harmonic function.
vii) If f = u + iv is analytic function, then the curves $u(x, y) = c_1$ cuts orthogonally the

curves $v(x, y) = c_2$.where c_1 and c_2 are constants.

Mappings:

1. i) Translation : w. = z + c

ii) Rotation: $w = e^{i\alpha}z$

iii) Contraction: w = kz

Where k is real and positive constant

iv) Rotation and magnification map: w = cz

Where c is complex constant

v) Inverse and reflection: $w = \frac{1}{z}$

vi) Linear transformation w = az + b.

2. **Conformal mapping:**

Suppose a mapping f(z) preserves angles both inmagnititude and direction between every pair of curves through a point then f(z) is said to be conformal at that point.

3. f(z) is said to be isogonalif it preserves themagnitudes of the angles but not the direction.,

4. At each point of the domain D where f(z) is analytic and $f'(z) \neq 0$, then the mapping w = f(z) is conformal.

5. **Necessary condition for w = f(z) to represent a conformal mapping :**

If the mapping w = f(z) is conformal then f(z) isan analytic function of z.

6. **Bilinear transformation:**

$w = \frac{az+b}{cz+d}$ad - bc$\neq$ 0 where a, b, c and d arecomplex constants is called a bileneartransformation.

7. Any bilinear transformation can be expressed as a product of translation, rotation, magnification orcontraction and inversion.

8. Any bilinear transformation maps the totality of circles and straight lines in the z-plane onto the totality of circles and straight lines in the w-plane.

9. **Fixed points:**

There are two points in the z-plane which will transform into themselves in w-plane. The fixed points of the bilinear transform $w = \frac{az+b}{cz+d}$ is given by $z = \frac{az+b}{cz+d}$

10. If z_1, z_2, z_3, z_4 are distinct points taken in order then the cross ratio of these points is

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

11. To find the bilinear transformation that maps z_1, z_2 and z_3 onto w_1, $w_2$and w_3 respectively is

$$\frac{w - w_1}{w_2 - w_1}\frac{w_2 - w_3}{w - w_3} = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1}$$

12. Normal formation of a bilinear transformation:

i) When there are two non-infinite fixed points, α, β then the bilinear transform is

$$\frac{w - \alpha}{w - \beta} = k\frac{z - \alpha}{z - \beta}$$

ii) Suppose α is a fixed point then the bileneartransform is $\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + k$

Complex integration:

1. Cauchy's integral theorem (or) Cauchy's fundamental theorem :

If a function f(z) is analytic at all points insideand on a closed contour C, then

$$\int_C f(z)dz = 0$$

2. **Cauchy's integral formula:**

If f(z) is analytic inside and on a closed curve C of a simply - connected domain D and if a is any point within D, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - a}$$

3. If f(z) is analytic inside the domain D bounded by C,then

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^3}$$

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$$

4. **Taylor's series:**

If a function f(z) is analytic inside a circle C with centre at a, then

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

5. **Maclaurin series:**

Put z = 0 in Taylor's series

$f(z) = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + ...$

$+ \frac{z^n}{n!} f^n(0) + ...$

6. **Laurent's series:**

If f(z) is analytic in the annulus (ring shaped region) between two concentric circles C_1 and C_2 with centre at a and radii R_1 and R_2 ($R_1>R_2$) for any point z in the annulus

$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$

Where, $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$ and

$$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)dz}{(z-a)^{1-n}}$$

7. **Singular points:**

If a function f(z) is not analytic at z = a, then it is called a singular point or singularity of f(z).

8. **Types of singularities:**

i) Isolated singularity:

Let z = a be a singular point of f(z). If there is no other singular point in the neighbourhood of z = a, then it is called isolated singular point.

ii) Removable singularity:

If the principal part of f(z) in its Laurent's series contains no term, then the singularity z = a is called removable singularity.

If z = a is a removable singularity then, $\lim_{z\to a} f(z)$ exists.

iii) Essential singularity:

If the principal part of f(z) in its Laurent's series contains Infinite number of terms, then z = ais called an essential singular point of f(z).

iv) Poles:

If we can find positive Integer n such that $\lim_{z\to a} (z-a)^n f(z) \neq 0$ then z = a is called a pole or order n for f(z).

A pole of order 1 is called a simple pole.

9. **Entire function:**

A f(z) is analytic everywhere in the finite plane (except at infinity) is called an entire function. Example: z,e^z, cosz

10. **Meromorphic function:**

A functionf(z) which is analytic everywhere in the finite plane except at finite number of poles is called a meromorphic function.

11. **Residues :**

The co-efficient b, of $\frac{1}{z-a}$ in the Laurent's seriesof f(z) is called the residue of f(z) at z = a

12. **Cauchy's Residue theorem:**

Let f(z) be single valued analytic function within and on a closed contour C. except at a finitenumber of poles$z_1, z_2, \ldots z_n$ within C and If R_1, R_1.........R_n be the residues of f(z) at these poles respectively then,

$\int_C f(z)dz = 2\pi i\ (R_1 + R_2 + \cdots + R_n)$

$= 2\pi i$ (Sum of the residues of poles within C)

13. **Formulae to find Residues:**

i) If z = a is a simple pole of f(z) then Residue of f(z) at z = a is $\lim_{z\to a}(z-a)\ f(z)$

ii) If z = a is a pole of order n then Residue of f(z) at z = a is

$\lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$

iii) If $f(z) = \frac{P(z)}{Q(z)}$ and z = a is a pole of orderone, then Residue of f(z) at z = a is $\frac{P(z)}{Q'(a)}$