Determinants and Matrices

1. The rank of a matrix A is said to be r if A satisfies the following conditions.
   i) There exists an r x r sub-matrix whose determinant is not zero.
   ii) The determinant of every (r+1) x (r+1) submatrix is zero.
2. Minor of a matrix A is the determinant formed by the elements of the matrix left after striking out certain rows and columns.
3. Consider the system of equations.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Let $A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}$

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} \cdots & a_{2n} & b_2 \\ a_{m1} & a_{m2} \cdots & a_{mn} & b_m \end{bmatrix}$$

Let the rank of A be R(A) and rank of [A, B] be R [A, B].

i) The system AX = B is consistent if and only if R (A) = R (A, B)

ii) If R(A) = R (A, B) = n (the number of unknowns), then the given system of equations is consistent and have unique solutions.

iii) If R(A) = R(A, B) < n then the given system of linear equations is consistent and have infinite number of solutions.

iv) If R(A)≠R(A, B), then the given system is not consistent (inconsistent) and have no solutions.

4. **Consider the system of equations :**

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$\Delta_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

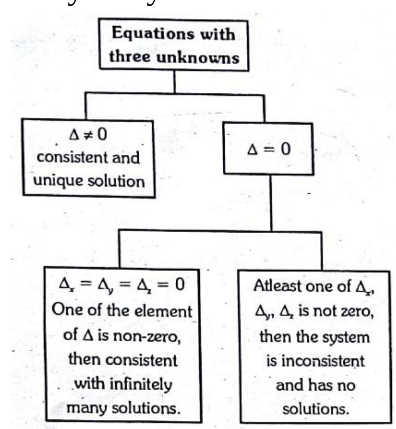
i) If $\Delta \neq 0$, the system is consistent and has unique solution.

Solutions are:

$x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$, $z = \frac{\Delta_z}{\Delta}$

ii) If $\Delta = 0$ and atleast one of the values of Δ_x, Δ_y, Δ_z is non-zero then the system has no solution.

iii) If $\Delta = 0$, $\Delta_x = \Delta_y = \Delta_z = 0$ and atleast one of the (2x2) minor of Δ is non-zero or atleast one of the element of Δ is non-zero, then the system is consistent and has infinitely many solutions.



5. Homogeneous system of linear equations :

A system of homogeneous linear equations are as follows:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_2$$

Let A = $\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & & \\ \vdots & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} & 0 \\ a_{21} & & & \\ \vdots & a_{22} \cdots & a_{2n} & 0 \\ a_{m1} & a_{m2} \cdots & a_{mn} & 0 \end{bmatrix}$$

Clearly, rank of A = rank of the augmented matrix [A, B].

i) The system of homogeneous equations is always consistent and obviously x_1 = x_2=......x_n= 0 is a trivial solution.

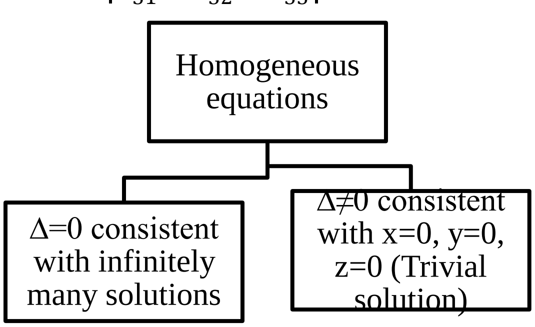
ii) If rank (A, B) = rank A = n (the number of unknowns) then the trivial solution is the unique solution.

iii) If rank (A, B) = rank A < n then the system has non-trivial solution. In this case |A| = 0

Consider the following system of homogeneous equations.

$$a_{11}x + a_{12}y + a_{13}z = 0$$
$$a_{21}x + a_{22}y + a_{23}z = 0$$
$$a_{31}x + a_{32}y + a_{33}z = 0$$

Let Δ = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$



6. Eigen values and Eigen vectors :

For n × n square matrix A the equation | A - λI | = 0 is said to be the characteristic equation. The n roots of | A - λI) = 0 are calledeigen values (characteristic roots, proper values(or) latent roots). Suppose $\lambda_1, \lambda_2, \ldots \lambda_n$ be theeigen values of A, corresponding to each value of λ_r the equation AX = λ_rX has a non-zero solution vector X_r. It is said to be eigen vector ofA corresponding toλ_r.

Properties of Eigen values:

i) Sum of eigen values is equal to the sum of the main diagonal elements of A (sum of eigen values = Trace of A)

ii) Product of eigen values of A = |A| (determinant of A)

iii) Every square matrix and its transpose have the same eigen values.

iv) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of A. Then,

a) A^{-1} has eigen values $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$

b) A ±kl has eigen values $\lambda_1 \pm k, \lambda_2 \pm k, \ldots\ldots \lambda_n \pm k$

c) A^2 has eigen values as $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$

d) A^m has eigen values $\lambda_1^m, \lambda_2^m, \ldots\ldots \lambda_n^m$

e) kA has eigen values as $k\lambda_1, k\lambda_2, \ldots k\lambda_n$

f) The eigen values of a triangular (upper or lower) matrix are the main diagonal elements.

Example:

Eigen values of $\begin{bmatrix} 2 & 4 & 5 \\ 0 & 7 & 8 \\ 0 & 0 & 1 \end{bmatrix}$ are 2, 7 and 1

Eigen values of $\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 6 & 5 & 7 \end{bmatrix}$ are 3, 4 and 7

v) If λ is an eigen values of an orthogonal matrix,then $\frac{1}{\lambda}$ is also its eigen value.

vi) The eigen values of a real symmetric matrix are real numbers.

vii) Eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal.

viii) If a real symmetric matrix of order 2 has equal eigen values then the matrix is a scalar matrix.

ix) If $\lambda_1, \lambda_2, \ldots \lambda_n$ be distinct eigen values ofa matrix A, then the corresponding eigenvectors $X_1, X_2, \ldots X_n$ form a linearlyindependent set.

x) Similar matrices:

A square matrix B of order n is called similar to a square matrix A of order n if B = S^{-1} AS for some non-singular matrix S of order n. Similar matrices have the same eigen values.

xi) Corresponding to a eigen values of A, there are different eigen roots of A. Corresponding to a eigen vector of a matrix, there exists only one eigen value.

7. Cayley - Hamilton Theorem :

Every square matrix satisfies its own characteristic equation.

Let A = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

If A is a square matrix of order 3, then its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where,

S_1 = Sum of the main diagonal elements i.e.,

$S_1 = a_{11} + a_{22} + a_{33}$

S_2 = Sum of the minors of the main diagonal elements.

$= \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

S_3= Determinant of A = |A|

If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Then, characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

Where,

S_1 = Sum of the main diagonal elements

$= a_{11} + a_{12}$

S_2= | A | (determinant of A)

Quadratic form:

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

Any quadratic form may be reduced to canonical form by means of a non-singular transformations.

Let Q = $ax^2 + by^2 + cz^2 + hxy + fyz + gzx$be a quadratic form.

The corresponding matrix is

$$\begin{bmatrix} \text{coeff. } x^2 & \frac{1}{2}\text{coeff. } xy & \frac{1}{2}\text{coeff. } xz \\ \frac{1}{2}\text{coeff. } xy & \text{coeff. } y^2 & \frac{1}{2}\text{coeff. } yz \\ \frac{1}{2}\text{coeff. } xz & \frac{1}{2}\text{coeff. } yz & \text{coeff. } z^2 \end{bmatrix}$$

$$= \begin{bmatrix} a & \frac{h}{2} & \frac{g}{2} \\ \frac{h}{2} & b & \frac{f}{2} \\ \frac{g}{2} & \frac{f}{2} & c \end{bmatrix}$$

Let X^{-1} AX be the quadratic form in n variables$x_1, x_2, \ldots x_n$

Let rank A = r

The number of positive square terms is called the index of the quadratic form and is denoted by s.

∴The number of non-positive terms (negative terms and zero terms) = r - s The difference between the positive square terms and the non-positive terms is called the signature of the quadratic form.

∴Signature = s — (r - s) = 2s - r

Let A = $[a_{ij}]$ be the matrix of the quadratic form.

Then,

$$D_1 = |a_{11}| = a_{11}$$

$$D_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\vdots$$

$$D_n = |A|$$

Nature	**Eigen value method**	**Principal minor method**	**Rank method** **s- index** **r - rank** **n - order of the matrix**
Positive definite	All are positive	$D_1, D_2, \ldots\ldots D_n$ all are positive	r = n and s = n
Negative Definite	All are negative	$D_1, D_3, D_{5\ldots\ldots}$ are negative $D_2, D_4, D_{6\ldots\ldots}$ are positive i.e., $(-1)^n D_n > 0$	r = n and 's = 0
Positive semi definite	All are positive and atleast one is zero	All are positive and atleast one Di = 0	r < n and s = r
Negative semi definite	All are negative and atleast one is zero	$D_1, D_3, \ldots\ldots$ are negative $D_2, D_4 \ldots\ldots$are positive atleast one is zero (or) $(-1)^n D_n \geq 0$	r < n and s = 0
Indefinite	Both positive and negative	Both positive and negative	All other cases