Calculus and Differential Equations

Differentiation Formulae:

S. No	y = f(x)	$\frac{dy}{dx} = f'(x)$
1.	x^n	nx^{n-1}
2.	e^x	e^x
3.	log x	$\frac{1}{x}$
4.	Sinx	cosx
5.	Cosx	$-\sin x$
6.	Tanx	$\sec^2 x$
7.	Cotx	$-\text{cosec}^2 x$
8.	secx	secx tanx
9.	cosecx	−cosecx cotx
10.	$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
11.	$\cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$
12.	$\tan^{-1}x$	$\frac{1}{1+x^2}$
13.	$\cot^{-1}x$	$\frac{-1}{1+x^2}$
14.	$\sec^{-1}x$	$\frac{1}{x\sqrt{x^2-1}}$
15.	$\text{cosec}^{-1}x$	$\frac{-1}{x\sqrt{x^2-1}}$
16.	sinhx	coshx
17.	coshx	sinhx
18.	tanhx	$\text{sech}^2 x$
19.	$\sinh^{-1}x$	$\frac{1}{\sqrt{1+x^2}}$
20.	$\cosh^{-1}x$	$\frac{1}{\sqrt{x^2-1}}$
21.	$\tanh^{-1}x$	$\frac{1}{1-x^2}$
22.	$\coth^{-1}x$	$\frac{1}{x^2-1}$

Product Rule:

$$\frac{d}{dx}(uv) = \frac{du}{dx}.v + u.\frac{dv}{dx}$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

Integration - Formulae:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
2. $\int \frac{dx}{x} = \log x + c$
3. $\int e^x dx = e^x + c$
4. $\int \sin x\, dx = -\cos x + c$
5. $\int \cos x\, dx = \sin x + c$
6. $\int \sec^2 x\, dx = \tan x + c$
7. $\int \text{cosec}^2 x\, dx = -\cot x + c$
8. $\int \sec x \tan x\, dx = \sec x + c$
9. $\int \text{cosec}x \cot x\, dx = -\text{cosec}x + c$
10. $\int \cosh x\, dx = \sinh x + c$
11. $\int \sinh x\, dx = \cosh x + c$

12. $\int \frac{dx}{1+x^2} = \tan^{-1}x + c$
13. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$
14. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x + c$
15. $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}x + c$
16. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c$
17. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
18. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$
19. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$
20. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$
21. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$
22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$
23. $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2}$
24. $\int \sqrt{a^2+x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2+x^2}}{2}$
25. $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$
26. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$
27. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

Partial Derivatives:

1. **Homogeneous function :**
   A function f(x, y, z) is called a homogeneous function of degree n if f(tx, ty, tz) = t^n f(x,y,z)
2. **Euler's theorem for homogeneous functions:**
   If f(x, y) is a homogeneous function of degree n then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.
3. **Remark:**
   If f(x_1, x_2,.............x_m) is a homogeneous functionof degree n then,
   $$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots x_m\frac{\partial f}{\partial x_m} = nf$$
4. **Extended Euler's theorem:**
   If f(x, y) is a homogeneous function of x, y of degree n then.
   $$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$
5. If u is a function of x and y, then,
   i) $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$
   ii) $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$
   If x and y are functions of t.
6. If f(x, y) = 0. Consider y as an implicit function of x.
   Then, $\frac{dy}{dx} = \frac{-fx}{fy}$

Jacobians:

1. If u(x, y) and v(x, y) are two functions then the Jacobian of u, v w.r.to x and y is
   $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$
   Also $\frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right)$
2. If u(x, y) and v(x, y,z) are functions of three variables x, y, z then.
   $$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$
3. $\frac{\partial(u,v)}{\partial(x,y)}.\frac{\partial(x,y)}{\partial(u,v)} = 1$
4. $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)}$

Jacobian of implicit functions:

5. If $f_1(x,y,u,v) = 0$
   $f_2(x,y,u,v) = 0$ then,

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\left(\frac{\partial(f_1,f_2)}{\partial(x,y)}\right)}{\left(\frac{\partial(f_1,f_2)}{\partial(u,v)}\right)}$$

If $f_1(x,y,z,u,v,w) = 0$

$f_2(x,y,z,u,v,w) = 0$

$f_3(x,y,z,u,v,w) = 0$

Then,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\left(\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\right)}{\left(\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}\right)}$$

Also $\frac{\partial(x,y,z)}{\partial(u,v,w} = (-1)^3 \frac{\left(\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}\right)}{\left(\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\right)}$

6. If $f_1(x,y,u,v) = 0$

$f_2(x,y,u,v) = 0$

Then,

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}; \frac{\partial u}{\partial y} = -\frac{\frac{\partial(f_1,f_2)}{\partial(y,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$\frac{\partial v}{\partial x} = -\frac{\frac{\partial(f_1,f_2)}{\partial(u,x)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}; \frac{\partial v}{\partial y} = -\frac{\frac{\partial(f_1,f_2)}{\partial(u,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

If $f_1(x,y,z,u,v,w) = 0$

then,

$\frac{\partial u}{\partial x} = \frac{\left(\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)}\right)}{\left(\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}\right)}; \frac{\partial v}{\partial x} = -\frac{\left(\frac{\partial(f_1,f_2,f_3)}{\partial(u,x,w)}\right)}{\left(\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}\right)}$ etc...

Maxima and Minima of two variables:

Let f(x, y) be a function of variables x and y.

Procedure to find maxima and minima of f(x,y) = 0

Put $\frac{\partial f}{\partial x} = 0; \frac{\partial f}{\partial y} = 0$

i.e., let $\frac{\partial f}{\partial x}(a, b) = 0; \frac{\partial f}{\partial y}(a, b) = 0$

Let $\frac{\partial^2 f}{\partial x^2}(a, b) = A$

$\frac{\partial^2 f}{\partial x\, \partial y}(a,b) = B$

$\frac{\partial^2 f}{\partial y^2}(a, b) = C$

i) f(a, b) is a maximum value if $AC - B^2 > 0$ and $A < 0$ (or $B < 0$)

ii) f(a, b) Is a minimum value if $AC - B^2 > 0$ and $A > 0$ (or $B > 0$)

iii) f(a, b) is not an extremum if $AC - B^2 < 0$

iv) If $AC - B^2 = 0$, we cannot decide whether (a, b)gives maximum or minimum.

Stationary value:

A function f(x, y) is said to be stationary at (a, b) if $\frac{\partial f}{\partial x}(a, b) = 0; \frac{\partial f}{\partial y}(a, b) = 0$

Stationary value is f(a, b)

To find stationary points put,

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$and solve these equations.

Differential Equations:

Consider the linear differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = f(x)$$

i.e., $(D^n y + a_1 D^{n-1} y + \cdots a_n)y$
$= f(x)$

The auxiliary equation is

$m_n + a_1 m^{n-1} + a_2 m^{n-2} + \dots\dots\dots a_n = 0$

Case (i):

If all the roots $m_1, m_2, \dots\dots m_n$ are real and differentthen,

C.F. = $Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x} + \dots\dots$

Case (ii):

If any two roots are equal say $m_1 = m_2 = m$ then, C.F. is $y = (Ax + B)e^{mx}$

Case (iii):

If any three roots are equal say $m_1 = m_2 = m_3 = m$, then, the C.F. is $y = (Ax^2 + Bx + C)e^{mx}$

Case (iv):

If the roots are imaginary say $m_1 = \alpha + i\beta$, $m_2 = a - i\beta$. Then C.F. is $y = e^{ax}(A \cos\beta x + B\sin\beta x)$

Particular integrals:

Consider $\phi(D)y = X$

If $X = e^{ax}$ then, $PI = \frac{e^{ax}}{\phi(a)}$

If $\phi(a) = 0$ then,

$PI = \frac{xe^{ax}}{\phi'(a)}$

If $\phi'(a) = 0$ then,

$PI = \frac{x^2e^{ax}}{\phi''(a)}$ provided $\phi''(a) \neq 0$

If $\phi''(a) = 0$ then,

$PI = \frac{x^3e^{ax}}{\phi'''(a)}$ provided $\phi'''(a) \neq 0$

If f(x)=sin ax (or) cosax

Then, $PI = \frac{\sin ax}{\phi(D)}$ (or) $\frac{\cos ax}{\phi(D)}$

Replace D^2 by $-a^2$

If $\phi(D) = 0$ then,

$PI = \frac{x \sin ax}{\phi'(D)}$ (or) $\frac{x \cos ax}{\phi'(D)}$

$$\frac{\sin \alpha x}{D^2 + \alpha^2} = \frac{-x \cos ax}{\alpha^2}$$

$$\frac{\cos \alpha x}{D^2 + \alpha^2} = \frac{x \sin ax}{\alpha^2}$$

Method of variation of parameters:

Consider $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X$

Let f_1 and f_2 be the solution of the auxilary equation

$m^2 + a_1 mx + a_2 = 0$

Then, C.F. = $C_1f_1 + C_2f_2$

Where, C_1 and C_2 are constants.

P.I. = $Pf_1 + Qf_2$

Where, $p = -\int \frac{f_2 \times dx}{f_1f_2' - f_1'f_2}$

$Q = \int \frac{f_1 \times dx}{f_1f_2' - f_1'f_2}$

Required solution y = C.F. + PI.